Hypersurfaces with many A_j-singularities: explicit constructions.

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Abstract. A construction of algebraic surfaces based on two types of simple arrangements of lines, containing the prototiles of substitution tilings, has been proposed recently. The surfaces are derived with the help of polynomials obtained from products of the lines generating the simple arrangements. One of the arrangements gives the generalizations of the Chebyshev polynomials known as folding polynomials. The other produces a family of polynomials having more critical points with the same critical values, which can also be used to derive hypersurfaces with many A_j-singularities.

Keywords: singularities, algebraic surfaces.

MSC: 14J17, 14J70

1. Introduction.

Algebraic hypersurfaces with many A_j-singularities have been studied in [17] by extending a construction of S.V. Chmutov [5], who used generalizations of the Chebyshev polynomials known as folding polynomials [15, 22]. Chmutov constructions produce surfaces having a high number of nodes or A_1-singularities, although for degrees $d = 6, 7, 8, 10, 12$ there are examples with a higher number [2, 11, 18, 21]. Nodal surfaces of several degrees have also been used to construct two-dimensional Calabi-Yau manifolds, also known as K3-surfaces. This is the case for the 600 nodal Sarti dodecic, a surface having a quotient which is a K3-surface [3]. New lower bounds for the number of singularities A_j on a hypersurface of degree d in the complex projective space $\mathbf{P}^n(\mathbf{C})$ are obtained for many cases in [17].

Two types of simple arrangements [14] of lines Σ_1, Σ_2, containing the prototiles of substitution tilings, have been introduced in [10]. Real variants of Chmutov surfaces [4] can be derived with the folding polynomials defined as products of lines in Σ_1. The mirror symmetric simple arrangements Σ_2, $\bar{\Sigma}_2$, with cyclic symmetries C_3, have one more triangle than Σ_1, and this property can be used to construct surfaces with more singularities. This is due to the fact that the corresponding polynomials have the same extreme values (-1 for all the minima and 8 for the maxima) in all the critical points with critical value of the same sign, which on the other hand coincide with those of the real folding polynomials constructed with Σ_1. The triangular shapes appearing in the simple arrangements are the prototiles of substitution tilings, derived with a general construction based on simplicial arrangements of lines [7]. A wide variety of tiling spaces can be defined having different types of topological invariants [8, 9], which are significant in several contexts, like the study of the energy gaps in the spectrum of the Schroedinger

equation with tiling equivariant potentials. In this work, the construction of new types of hypersurfaces with many A_j-singularities is done with the polynomials associated with Σ_2 and $\bar{\Sigma}_2$. Mathematica [23], Singular [13] and Surfer [6, 16] computing and geometric visualization tools are used.

2. Surfaces with many A_1-singularities

Let $T_d^C(w) \in \mathbf{R}[w]$ be the Chebyshev polynomial of degree d with critical values -1,+1 and $F_d(u,v)$ the folding polynomials associated to the system of roots $\mathbf{A}_2$. An explicit formula for the folding polynomials of degree d, which are symmetric ($F_d(u,v) = F_d(v,u)$) with real coefficients, is $F_d(u,v) = 2 + j_d(u,v) + j_d(v,u)$, where

$$j_d(u,v) = \det \begin{pmatrix} u & 1 & 0 & 0 & 0 & 0 & ... & 0 \\ 2v & u & 1 & 0 & 0 & 0 & ... & 0 \\ 3 & v & u & 1 & 0 & 0 & ... & 0 \\ 0 & 1 & v & u & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & v & u & 1 & 0.. & 0 \\ ... & ... & ... & ... & ... & ... & ... & 0 \\ ... & ... & ... & 0 & 1 & v & u & 1 \\ 0 & ... & ... & 0 & 0 & 1 & v & u \end{pmatrix}$$

Chmutov constructs surfaces $Ch_d(u,v,w) := F_d(u,v) + (T_d^C(w)+1)/2$ having many nodes. In [4] it is shown that the Chmutov construction can be adapted to give only real singularities. The real folding polynomials $F_d(x,y)$ are obtained when $u = x+iy, v = u^* = x - iy$. The plane curve $F_d(x,y)$ is a product of d lines in simple arrangements Σ_1 having critical points with only three different critical values: 0,-1 and 8. The surface $Ch_d(x,y,z)$ is singular exactly at the points where the critical values ζ of $F_d(x,y)$ and $(T_d^C(z)+1)/2$ are either both zero, or one is -1 and the other +1.

The polynomials $F_d(x,y)$ have $\binom{d}{2}$ real critical points with value $\zeta = 0$ and, when $d \in \mathbf{Z}_3$, $\frac{1}{3}d^2 - d$ real critical points with $\zeta = -1$. The other critical points also have real coordinates and critical value $\zeta = 8$. The number of maxima can then be obtained by applying the following

Lemma 1. [19, 4] Let f be a real simple line arrangement consisting in $d \geq 3$ lines. f has exactly $\binom{d-1}{2}$ bounded open cells each of which contains exactly one critical point. All the critical points of f are non-degenerate.

It is possible to construct polynomials for some degrees with one more critical point with $\zeta = -1$. For $m = 3q, q = 1,2,3,...$ the simple arrangement $\bar{\Sigma}_2$ consists in the lines $\bar{L}_{k,m}(x,y) = 0$, with

$$\bar{L}_{k,m}(x,y) := y - (\cos\frac{k\pi}{3m} - x)\tan\frac{k\pi}{6m} - \sin\frac{k\pi}{3m} \tag{1}$$

where $k \in \mathbf{Z}$ and $\bar{L}_{3m,m}(x,y) = 0$ is interpreted as the line $x = -1$. The polynomials obtained with $\bar{L}_{k,m}(x,y)$ are defined as

$$\bar{J}_m^C(x,y) := 3^{\frac{1-(-1)^m}{4}}(-1)^{\lfloor\frac{q+1}{2}\rfloor+1}\prod_{\nu=0}^{m-1}\bar{L}_{6\nu+1,m}(x,y) \in \mathbf{R}[x,y] \tag{2}$$

and $J_m^C(x,y) = \bar{J}_m^C(x,-y)$. They have only three different critical values: 0,-1, 8, and by changing the variables $x = (u+v)/2, y = i(v-u)/2$ in $J_m^C(x,y)$, we get

$$J_m^C(u,v) = -1 + j_m^C(u,v) + j_m^{C*}(u,v)$$

where $j_m^C(u,v)$ has now complex coefficients. The polynomials corresponding to $d = 3, 6, 9$, are

$$j_3^C(u,v) = -bu^3 + \frac{3}{2}uv,$$

$$j_6^C(u,v) = (2-8b)u^3 - bu^6 + 6uv + 6bu^4v - \frac{9}{2}u^2v^2,$$

$$j_9^C(u,v) = (9-27b)u^3 - 9bu^6 - bu^9 + \frac{27}{2}uv - (9-54b)u^4v + 9bu^7v - 27u^2v^2 - 27bu^5v^2 + 15u^3v^3,$$

with $b = e^{-i\frac{\pi}{3}}$. In contrast with $F_d(u,v)$, the polynomials $J_m^C(v,u)$ are not symmetric and $\bar{J}_m^C(u,v) := J_m^C(u^*,v^*) = J_m^C(v,u)$. The surfaces with affine equations [10]

$$Q_m^C(x,y,z) := J_m^C(x,y) + \frac{(-1)^{m+1}}{4}(J_m^C(z,0) - 1 + (-1)^{m+1}2) = 0 \quad (3)$$

have a number of real nodes, or A_1-singularities, higher than the real variants of the Chmutov surfaces with the same degree. The first surface in the series defined by Eq.(3) is equivalent to the Cayley cubic, a well known example having four ordinary double points, the maximum possible number in degree 3. The Chmutov surface of the same degree has three nodes.

3. Hypersurfaces with many A_j-singularities.

An A_j-singularity on a hypersurface in $\mathbf{P}^n(\mathbf{C})$ has the local equation $x_0^{j+1}+x_1^2+x_2^2+...+x_{n-1}^2 = 0$. The polynomials $J_m^C(x,y)$ and $\bar{J}_m^C(x,y)$ can be used for the construction of hypersurfaces with many A_j-singularities.

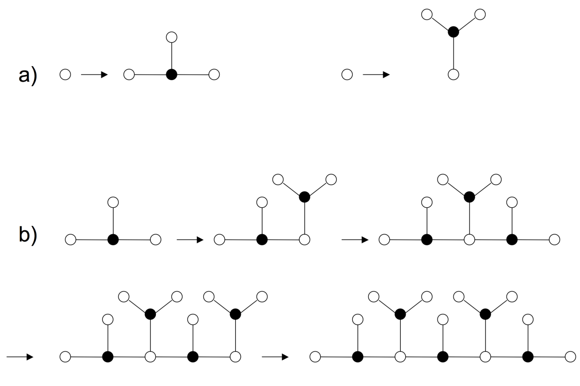


Figure 1. Plane trees for Belyi polynomials.(a)The two types of substitution rules. (b) Plane trees corresponding to $\bar{B}_{3q}^2(z)$, $q = 1,2,3,4,5$, obtained by applying alternatively the substitutions to the right most vertex.

A polynomial in one variable with no more than two different critical values is called a Belyi polynomial. The proof of its existence is based on the theory of Dessins d´Enfants. A graph

without cycles with a prescribed cyclic order of the edges adjacent to each vertex is called a plane tree [1]. A plane tree has a natural bicoloring of the vertices. For any plane tree, there exists a Belyi polynomial whose critical points have the multiplicities given by the number of edges adjacent to the vertices minus one and viceversa. Degree-d Chebyshev polynomials are the Belyi polynomials whose plane tree has the form of the Dynkin diagram of the Lie algebra $\mathbf{A}_{d+1}$. The plane trees of a family of Belyi polynomials that we denote by $\bar{B}^2_{3q}(z)$ (see Appendix B), can be obtained by a substitution process as indicated in Fig.1. After the first step, which is given in the left part of Fig.1a, we apply alternatively to the right most white vertex, the two substitution rules given in Fig.1a. The next four steps, corresponding to $q = 2, 3, 4, 5$ are shown in Fig.1b. The generalization of the Chmutov´s construction given in [17] for the study of singularities of type A_j in $\mathbf{P}^3(\mathbf{C})$ consists in considering surfaces with affine equations

$$F_d(x, y) + G^j_d(z) = 0 \tag{4}$$

where $G^j_d(z)$ are Belyi polynomials. In general there is no explicit formula for them, and they can be computed only for low degree d by using Groebner basis. However, in some cases, there are explicit expressions that can be obtained from classical Jacobi polynomials $P^{l,m}_k(z)$ [20] by studying the three-parameter family of polynomials

$$G_{a,b,c}(z) = z^a P^{a/c,-b}_{b-1}(1 - 2z)^c \tag{5}$$

The order of a zero z_0 of $\frac{dP(z)}{dz}$, which is a critical point of $P(z)$ with critical value $\zeta = P(z_0)$, is called its multipicity ν (all the derivatives of $P(z)$ up to order ν vanish at z_0). The zeros of $\frac{dG_{a,b,c}}{dz}$ with critical value $\zeta = 0$ have orders $\nu = a - 1, c - 1, ..., c - 1$, while the unique remaining root $r = 1$ has $\nu = b - 1$ with critical value $\zeta = 1$.

We denote by μ_{A_j} the number of A_j-singularities ($\mu_{A_j}(d)$ if we want to specify the degree d). If we replace $F_{3q}(x, y)$ by $J^C_{3q}(x, y)$ or $\bar{J}^C_{3q}(x, y)$ in Eq.(4) we can get in some cases surfaces with a higher μ_{A_j}. In what follows, we use $G_{a,b,c}(z)$ in order to give explicit expressions for some singular surfaces. Even when the number of singularities of a given type is equal to the one obtained in [17], the surfaces show a higher number of other types. The lines $\bar{L}_{k,m}(x, y) = 0$ and $\bar{L}_{l,m}(x, y) = 0$, where

$$\bar{L}_{k,m}(x, y) = y + t_{k,m}x - B_{k,m} \tag{6}$$

with $t_{k,m} = \tan\frac{k\pi}{6m}, B_{k,m} = (3t_{k,m} - t^3_{k,m})/(1 + t^2_{k,m})$, intersect in the point

$$\left(\frac{B_{k,m} - B_{l,m}}{t_{k,m} - t_{l,m}}, \frac{B_{l,m}t_{k,m} - B_{k,m}t_{l,m}}{t_{k,m} - t_{l,m}}\right) \tag{7}$$

which is denoted, for a given m, by $p_{k\cap l}$.

Proposition 1. The surfaces $J^C_6(x, y) + G_{2,2,4}(z) = 0$ in $\mathbf{P}^3(\mathbf{R})$ have $\mu_{A_1} = 22$ and $\mu_{A_3} = 15$. In $\mathbf{P}^4(\mathbf{R})$ the three-fold $J^C_6(x_0, x_1) - J^C_6(x_2, x_3) = 0$ has $\mu_{A_1} = 283$.

Proof. The points corresponding to the minima of $J^C_6(x, y)$ are, up to three-fold rotation, $p_{0\cap 6}, p_{0\cap 12}, p_{2\cap 8}$ with critical value $\zeta = -1$ and the maximum with critical value $\zeta = 8$ is in $p_{4\cap 10}$. Except for the minima situated at the origin ($p_{0\cap 12}$ in this case), the action of the cyclic group C_3 gives the remaining distinct critical points for this and the examples that follows. By representing the points in the plane by complex numbers, we find that the minima are in

$$0, \alpha, r\alpha, r^2\alpha, 2, 2r, 2r^2$$

where $\alpha = e^{i\frac{\pi}{9}}, r = e^{i\frac{2\pi}{3}}$. The maxima correspond to the conjugates of the minima which are not themselves minima. In this case the maxima are in

$$\alpha^*, r^*\alpha^*, r^{*2}\alpha^*$$

and $Q_6^C(x, y, z)$ has Tjurina number $\tau = 59$ (see Appendix A). The polynomial $G_{2,2,4}(z) = \frac{(-3+z)^4 z^2}{16}$ has critical points $z = 0, 3$ (multiplicity $\nu = 3$) with $\zeta = 0$ and $z = 1$ with $\zeta = 1$. Although 15 is the best known lower bound of μ_{A_3} for a sextic in $\mathbf{P}^3(\mathbf{C})$, the surface presented here has higher μ_{A_1} and the singularities are real (Fig.2a).

Proposition 2. In $\mathbf{P}^4(\mathbf{R})$ the hypersurface $J_9^C(x_0, x_1) - J_9^C(x_2, x_3) = 0$ has $\mu_{A_1} = 1738$. The following polynomials define surfaces of type $J_9^C(x, y) + G_{a,b,c}(z) = 0$ in $\mathbf{P}^3(\mathbf{K})$ with the indicated number of singularities:

2.1.- $G_{3,3,3}(z)$; $\mathbf{K} = \mathbf{C}$; $\mu_{A_2} = 127$.

2.2.- $G_{1,3,4}(z)$; $\mathbf{K} = \mathbf{C}$; $\mu_{A_3} = 72, \mu_{A_2} = 19$.

2.3.- $G_{3,2,6}(z)$; $\mathbf{K} = \mathbf{R}$; $\mu_{A_5} = \mu_{A_2} = 36, \mu_{A_1} = 19$.

The maximum number of cusps and real A_4-singularities for a nonic surface satisfy $\mu_{A_2}(9) \geq 127, \mu_{A_4}(9) \geq 55$.

Proof. The result for the hypersurface in $\mathbf{P}^4(\mathbf{R})$ is obtained having in mind that the minima of $J_9^C(x, y)$ are in $p_{0\cap 6}, p_{0\cap 12}, p_{0\cap 18}, p_{0\cap 24}, p_{2\cap 8}, p_{2\cap 14}, p_{8\cap 14}$ with $\zeta = -1$ and the maxima with $\zeta = 8$ are situated in $p_{4\cap 10}, p_{4\cap 16}, p_{10\cap 16}$. The number of singularities of the surfaces in $\mathbf{P}^3(\mathbf{K})$ can be computed by using the following properties:

1.- $G_{3,3,3}(z) = z^3\left(3 - 3z + z^2\right)^3$ has critical points $z = 0, \frac{3 \pm i\sqrt{3}}{2}$ with $\zeta = 0$ and $z = 1$ with $\zeta = 1$, all of them with $\nu = 2$.

2.- $G_{1,3,4}(z) = z\left(\frac{45}{32} - \frac{9z}{16} + \frac{5z^2}{32}\right)^4$ has critical points $z = \frac{9 \pm 12i}{5}$ $(\nu = 3)$ with $\zeta = 0$ and $z = 1$ $(\nu = 2)$ with $\zeta = 1$.

3.- $G_{3,2,6}(z) = \frac{(-3+z)^6 z^3}{64}$ has critical points $z = 0$ $(\nu = 2)$, 3 $(\nu = 5)$ with $\zeta = 0$ and $z = 1$ with $\zeta = 1$. The surface $J_9^C(x, y) + G_{3,2,6}(z) = 0$ is represented in Fig.2b.

By using other Belyi polynomials instead of $G_{3,3,3}(z)$, one gets surfaces with $\mu_{A_2} = 110$ (see Appendix B). The existence of the surface $J_9^C(x, y) + G_{3,3,3}(z) = 0$ increases in one cusp the known lower bound of μ_{A_2} for a nonic [17]. The Belyi polynomial $\bar{B}_9^4(z)$ given in Appendix B produces a nonic with $\mu_{A_4} = 55$.

Proposition 3. The three-fold $J_{12}^C(x_0, x_1) - J_{12}^C(x_2, x_3) = 0$ in $\mathbf{P}^4(\mathbf{R})$ has $\mu_{A_1} = 6049$. The number of singularities of $J_{12}^C(x, y) + G_{a,b,c}(z) = 0$ in $\mathbf{P}^3(\mathbf{K})$ is

3.1.- $\mu_{A_4} = 132, \mu_{A_2} = 37, \mu_{A_1} = 66$ for $a = 2, b = 3, c = 5$; $\mathbf{K} = \mathbf{C}$.

3.2.- $\mu_{A_7} = \mu_{A_3} = 66, \mu_{A_1} = 37$ for $a = 4, b = 2, c = 8$; $\mathbf{K} = \mathbf{R}$.

The following lower bound is valid for dodecic surfaces: $\mu_{A_2}(12) \geq 301$.

Proof. For degree-12 we have

$$j_{12}^C(u, v) = (24 - 64b)u^3 - (2 + 40b)u^6 - 12bu^9 - bu^{12} + 24uv - (60 - 240b)u^4v + 96bu^7v + 12bu^{10}v$$

$$-90u^2v^2 - (36 - 288b)u^5v^2 - 54bu^8v^2 + 120u^3v^3 - 28bu^6v^3 - \frac{105}{2}u^4v^4,$$

with $b = e^{-i\frac{\pi}{3}}$. We denote by $S[A]$ the 2-combinations or subsets of two elements $\{k, l\}$ taken from the set A. The values $\{k, l\}$ in $p_{k\cap l}$ for the minima of $J_{12}^C(x, y)$ with $\zeta = -1$

are $\{6,12\}$, $\{66,60\}$,$\{0,6n\}, n=1,2,3,4,5$ together with the ones specified by the elements of $S[\{2,8,14,20\}]$. The maxima with $\zeta=8$ are obtained from $S[\{4,10,16,22\}]$. Now we have:

1.-$G_{2,3,5}(z)=z^2\left(\frac{42}{25}-\frac{24\,z}{25}+\frac{7\,z^2}{25}\right)^5$ has critical points $z=0, \frac{12\pm i5\sqrt{6}}{7}$ $(\nu=4)$ with $\zeta=0$ and $z=1$ $(\nu=2)$ with $\zeta=1$.

2.-$G_{4,2,8}(z)=\frac{(-3+z)^8\,z^4}{256}$ has critical points $z=0$ $(\nu=3)$,3 $(\nu=7)$ with $\zeta=0$ and $z=1$ with $\zeta=1$.

The existence of $J^C_{12}(x,y)$ and the Belyi polynomial with the plane tree as in Fig.1b, gives a surface with $\mu_{A_2}(12)=301$.

Proposition 4. In $\mathbf{P}^4(\mathbf{R})$ the three-fold $J^C_{15}(x_0,x_1)-J^C_{15}(x_2,x_3)=0$ has $\mu_{A_1}=15646$. The number of singularities of $J^C_{15}(x,y)+G_{a,b,c}(z)=0$ in $\mathbf{P}^3(\mathbf{K})$ is

4.1.- $\mu_{A_3}=376, \mu_{A_2}=61$ for $a=3,b=4,c=4$; $\mathbf{K}=\mathbf{C}$.

4.2.- $\mu_{A_5}=210, \mu_{A_2}=166$ for $a=3,b=3,c=6$; $\mathbf{K}=\mathbf{C}$.

4.3.- $\mu_{A_9}=\mu_{A_4}=315, \mu_{A_1}=61$ for $a=5,b=2,c=10$; $\mathbf{K}=\mathbf{R}$.

We have the following lower bounds for 15-degree surfaces: $\mu_{A_2}(15)\geq 647, \mu_{A_3}(15)\geq 376$.

Proof. In this case

$$j^C_{15}(u,v)=(50-125b)u^3-(15+125b)u^6-75bu^9-15bu^{12}-bu^{15}+\frac{75}{2}uv-(225-750b)u^4v$$
$$+(15+525b)u^7v+165bu^{10}v+15bu^{13}v-225u^2v^2+(315-1575b)u^5v^2-675bu^8v^2-90bu^{11}v^2$$
$$+525u^3v^3-(140-1400b)u^6v^3+275bu^9v^3-525u^4v^4-450bu^7v^4+189u^5v^5,$$

with $b=e^{-i\frac{\pi}{3}}$. The subsets $\{k,l\}$ associated to the minima of $J^C_{15}(x,y)$ with $\zeta=-1$ are $\{6,12\}, \{6,18\}, \{84,78\}, \{84,72\}$,$\{0,6n\}, n=1,2,...,7$, as well as those given by the elements of $S[\{2,8,14,20,26\}]$. The maxima with $\zeta=8$ are obtained from $S[\{4,10,16,22,28\}]$. The polynomials $G_{a,b,c}(z)$ considered here have the following properties:

1.-$G_{3,4,4}(z)=z^3\left(\frac{385}{128}-\frac{495\,z}{128}+\frac{315\,z^2}{128}-\frac{77\,z^3}{128}\right)^4$ has critical points $z=0$ $(\nu=2)$, $z\approx 2.20309$ $(\nu=3)$, $z\approx 0.94391\pm i1.17413$ $(\nu=3)$ with $\zeta=0$ and $z=1$ $(\nu=3)$ with $\zeta=1$.

2.-$G_{3,3,6}(z)=z^3\left(\frac{15}{8}-\frac{5\,z}{4}+\frac{3\,z^2}{8}\right)^6$ has critical points $z=0$ $(\nu=2)$, $z=\frac{3\pm i2\sqrt{5}}{7}$ $(\nu=5)$ with $\zeta=0$ and $z=1$ $(\nu=2)$ with $\zeta=1$.

3.-$G_{5,2,10}(z)=\frac{(-3+z)^{10}\,z^5}{1024}$ has critical points $z=0$ $(\nu=4)$, 3 $(\nu=9)$ with $\zeta=0$ and $z=1$ with $\zeta=1$.

The Belyi polynomial $\bar{B}^2_{15}(z)$ associated with the plane tree in Fig.1b, together with $J^C_{15}(x,y)$ gives a surface with $\mu_{A_2}(15)=647$. We notice that $J^C_{15}(x,y)+G_{3,4,4}(z)=0$ has one more A_3-singularity than the surfaces with the same degree having the highest known μ_{A_3} [17]. Continuing along this lines, degree-$3q$ surfaces in $\mathbf{P}^3(\mathbf{R})$ with singularities of type $A_{2q-1}, q=1,2,...$ can be obtained by adding $G_{q,2,2q}(z)$ to $J^C_m(x,y)$ or $\bar{J}^C_m(x,y)$.

Nodal hypersurfaces in $\mathbf{P}^n(\mathbf{R})$ can be constructed also with the help of $J^C_m(x,y)$:

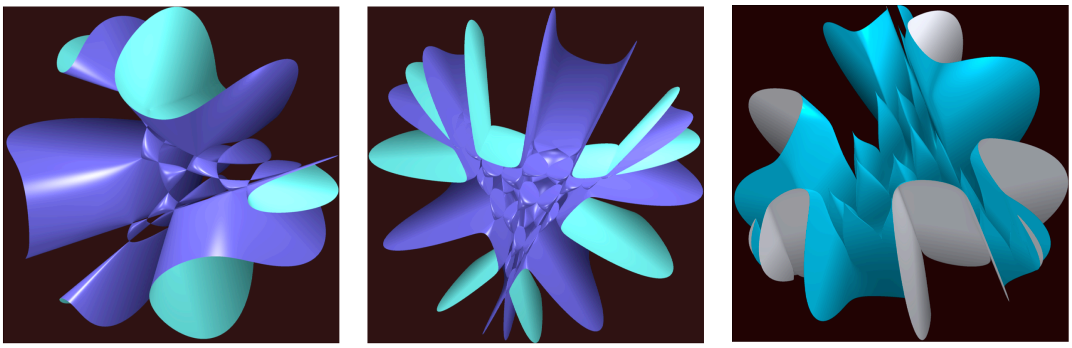
$$\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}(-1)^kJ^C_m(x_{2k},x_{2k+1})=\frac{(-1)^{m+n}+(-1)^{m+1}}{8}(J^C_m(x_{n-1},0)-1+(-1)^{m+1}2) \tag{8}$$

For instance $J^C_6(x_0,x_1)-J^C_6(x_2,x_3)=\frac{1}{4}(3-J^C_6(x_4,0))$ in $\mathbf{P}^5(\mathbf{R})$ has 1059 nodes. These can be computed having in mind the results in Prop.1 and the fact that $\frac{1}{4}(3-J^C_6(x_4,0))$ has 3 critical points with $\zeta=0$ and 2 critical points with $\zeta=1$.

The methods given above can be extended in order to construct hypersurfaces in $\mathbf{P}^n(\mathbf{C}), n > 4$ with many A_j-singularities, by using $G_{a,b,c}(x_{n-1})$ instead of the right-hand side of Eq.(8). In this way we get, for example, the hypersurface $J_6^C(x_0, x_1) - J_6^C(x_2, x_3) = G_{2,2,4}(x_4)$ in $\mathbf{P}^5(\mathbf{R})$ with 388 nodes and 283 A_3-singularities.

There is a dynamical formulation in terms of a uniparametric family of line configurations [9] which can be used to get deformations of the surfaces, where some singularities disappear and others arise. It gives a way to transform the surfaces based on Σ_1 into the surfaces based on Σ_2 and $\bar{\Sigma}_2$, representing a topology change that would correspond to a kind of phase transition in a physical context.

We have studied hypersurfaces with a high number of singularities, improving the known lower bounds in some cases. Open questions that should be addressed are the search of maximal simple arrangements of lines giving polynomials with the same critical values, and deformations of the surfaces to get others with new types of singularities.



a) b) c)

Figure 2. Examples with real singularities: a) A degree 6 surface with $\mu_{A_1} = 22$ and $\mu_{A_3} = 15$.b) A nonic surface with $\mu_{A_1} = 19$ and $\mu_{A_2} = \mu_{A_5} = 36$. c) A nonic with $\mu_{A_4} = 55$.

4. APPENDIX A: Milnor and Tjurina numbers.

Let f be a holomorphic complex function germ at a given point. By O we denote the ring of function germs and by

$$\mathrm{J}_f := < \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots \frac{\partial f}{\partial x_n} >$$

the jacobian ideal of f. The Milnor algebra M of f is given by the quotient algebra $O/\mathrm{J_f}$ and the Milnor number is given by the complex dimension of M. The Tjurina number τ is the dimension of the algebra obtained by replacing $\mathrm{J_f}$ by the ring

$$< f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots \frac{\partial f}{\partial x_n} >$$

.

In what follows we use SINGULAR computer algebra system [12, 13] for checking the Tjurina and Milnor numbers. The basering is a polynomial ring with three variables over the algebraic number field $\mathbf{Q}(a)$, $a = \sqrt{3}$. Lexicographical and degree reverse lexicographical monomial ordering are denoted by lp, dp, and vdim(std(I)) is the vector space dimension of the ring modulo de ideal I. Short input is used (e.g. $3x^2 - x^3$ is denoted by $3x2 - x3$). The result of the code

```
LIB "sing.lib";
ring R = (0,a),(x,y,z),dp; //affine ring, , 0 is the characteristic of the ground field
minpoly=a2-3;//the minimal polynomial of a
poly f = -1+3x2-x3-3ax2y+3y2+3xy2+ay3+(3z2-z3)/4;
ideal sl = jacob(f),f; //the singular locus
vdim(std(sl)); //Total Tjurina number
```

is $\tau = 4$, which is the number of singularities of the cubic $Q_3^C(x, y, z) = 0$. For $Q_d^C(x, y, z), d = 6, 9, 12, 15$ we get $\tau = 59, 220, 581, 1162$.

We can also use the built-in commands milnor and tjurina. For instance we can check that all the critical points of $J_9^C(x, y)$ are non degenerated (Lema 1):

```
ring R = (0,alpha),(x,y),dp;
minpoly=a2-3;
poly f = -1+27x2-9x3-54x4+36x5+21x6-27x7+9x8-x9- 81ax2y+162ax4y-54ax5y-81ax6y
+54ax7y-9ax8y +27y2+27xy2-108x2y2-72x3y2+225x4y2+27x5y2-126x6y2+36x7y2
+27ay3+108ax2y3+180ax3y3-135ax4y3-126ax5y3+84ax6y3- 54y4-108xy4-45x2y4
+135x3y4-126x5y4 -54ay5-54axy5-27ax2y5-126ax3y5-126ax4y5 +39y6+81xy6+126x2y6
+84x3y6+ 27ay7+54axy7+36ax2y7 -9y8-9xy8 -ay9;
milnor(f);
ideal j = jacob(f); // the critical locus of f
poly h = det(jacob(j)); //determinant of the Hessian of f
ideal nn = j,h;
vdim(std(nn));
```

gives a Milnor number of 64, and vdim(std(nn))=0. For the sextic with non-nodes, the code

```
ring R = (0,a),(x,y,z),dp;
minpoly=a2-3;
poly f =-1+12x2-4x3-9x4+6x5-x6-24ax2y+18ax4y-6ax5y+12y2+12xy2-18x2y2
-12x3y2+15x4y2+8ay3+12ax2y3+20ax3y3-9y4-18xy4-15x2y4-6ay5-6axy5+y6
+(z6-12z5+54z4-108z3+81z2)/16;
tjurina(f);
```

gives $\tau = 67$, as expected from Prop.1. In a similar way, if we analyze $f = J_9^C(x, y) + G_{3,3,3}(z)$, we get $\tau = 254$, which corresponds to a surface with 127 cusps as in Prop.2.

5. APPENDIX B: Belyi polynomials.

In this appendix we obtain several Belyi polynomials. Related with A_2-singularities for nonics are $B_9^2(z)$ and $\bar{B}_9^2(z)$. The roots of $\frac{dB_9^2(z)}{dz}$ with critical value $\zeta = -1$ are denoted by a, b, while $0, u$ have critical value $\zeta = 1$, all of them with multiplicity $\nu = 2$. We have $\frac{dB_9^2(z)}{dz} = (z - a)^2(z - b)^2(z - u)^2 z^2$, with $B_9^2(0) = 1$. In order to get the values of the critical points we use the following code

```
ring R = 0,(a,b,u),lp;
poly f1 = 2520+5a9-15a8b+12a7b2-15a8u+48a7bu-42a6b2u+12a7u2-42a6bu2+42a5b2u2;
poly f2 = 2520+12a2b7-15ab8+5b9-42a2b6u+48ab7u-15b8u+42a2b5u2-42ab6u2+12b7u2;
poly f3 = 42a2b2-42a2bu-42ab2u+12a2u2+48abu2+12b2u2-15au3-15bu3+5u4;
ideal I = f1, f2, f3;
ideal GI = groebner(I);
GI;
```

The Groebner basis has four elements: $GI_1 = GI_1(u), GI_2 = GI_2(u,b), GI_3 = GI_3(u,b), GI_4 = GI_4(u,b,a)$. The polynomials GI_1, GI_2, with degrees 180,172, have the common factor $u^{36} + 135413275557888$. If we take one of its roots, for instance $u = 2^{\frac{5}{9}} 3^{\frac{17}{36}} e^{\frac{i\pi}{36}}$, then a, b are the two complex roots of

$$z^2 - u2^{-\frac{3}{2}}(2^{\frac{3}{2}} - 3^{-\frac{1}{4}} - 3^{\frac{1}{4}} + i(3^{-\frac{1}{4}} - 3^{\frac{1}{4}}))z + \frac{u^2}{24}(6 - 2^{\frac{1}{2}}3^{\frac{5}{4}} - 2^{\frac{1}{2}}3^{\frac{3}{4}} + i(-2^{\frac{1}{2}}3^{\frac{5}{4}} - 3^{\frac{1}{2}}2 + 2^{\frac{1}{2}}3^{\frac{3}{4}}))$$

In a similar way we can get a polynomial $\bar{B}_9^2(z)$ with three critical points with critical value -1, corresponding to the black vertices in Fig.1b, $q = 3$. The roots of $\frac{d\bar{B}_9^2(z)}{dz}$ with $\zeta = -1$ are denoted by $a, b, 0$, while u has $\zeta = 1$. We obtain in this case $u^9 + 18 = 0$, and a, b are the roots of $z^2 - 3uz + 3u^2 = 0$. We notice that $\bar{B}_9^2(z)$ has complex coefficients even if we take $u \in \mathbf{R}$. It has the same planar tree as $G_{3,3,3}(z)$ in Prop.2, but the polynomial derived from the classical Jacobi polynomial is simpler.

The surfaces $J_9^C(x,y) + G_9^2(z) = 0$, with $G_9^2(z) = (B_9^2(z) + 1)/2$ and $G_9^2(z) = (\bar{B}_9^2(z) + 1)/2$, have 110 and 127 cusps respectively. The surfaces $J_{3q}^C(x,y) + (\bar{B}_{3q}^2(z) + 1)/2 = 0$ have $\lfloor \frac{3q-1}{2} \rfloor - \lfloor \frac{3q}{3} \rfloor$ more cusps than those constructed in [17] with $F_d(x,y)$.

Other lower bounds can be improved by using $J_m^C(x,y)$. For $m = 9, j = 4$ we define $\bar{B}_9^4(z)$ in such a way that $\frac{d\bar{B}_9^4(z)}{dz} = (z-a)^4(z-b)^4$, with $\bar{B}_9^4(a) = 1, \bar{B}_9^4(b) = -1$. The Groebner basis has 2 elements and if we take the real root of $315 + 128b^9 = 0$, then $a = -b$. In contrast to the cases studied above we find a solution for $\bar{B}_9^4(z)$ with real coefficients. The surface $J_9^C(x,y) + (\bar{B}_9^4(z) + 1)/2 = 0$, where the normalized Belyi polynomial is

$$\frac{1}{80640}(40320 + 6^{7/9}35^{8/9}945z - 6^{1/3}35^{2/3}5040z^3 + 6^{8/9}35^{4/9}3024z^5 - 6^{4/9}35^{2/9}5760z^7 + 4480z^9),$$

has 55 real singularities of type A_4 (Fig.2c).

6. References

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